

The University of Chicago

A GENERALIZED FORM OF THE
PROBLEM OF BOLZA

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1937

By
CARL HERBERT DENBOW

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Introduction. The problem studied in this dissertation may be described as one having the form of a problem of Bolza in the calculus of variations, except that the function to be minimized depends, not only on the coordinates of the end-points of an admissible curve, but also on the coordinates of variable intermediate points on this curve. The coordinates of the end-points and the intermediate points are further restricted to satisfy certain side conditions. This problem may be transformed into an equivalent problem of Bolza by a transformation due to W. T. Reid and L. M. Graves. In the present paper it will be shown how the existing theory of the problem of Bolza can, with slight alterations, be used to derive necessary conditions and sufficient conditions for the original problem. In particular it is shown that a minimizing arc must satisfy the Euler-Lagrange equations, certain transversality and corner conditions, the Weierstrass and Clebsch conditions, and analogues of the generalized Jacobi conditions for a problem of Bolza. These conditions, suitably strengthened, are shown to be sufficient to insure a minimum.

The exceptional cases where, for a minimizing set, two or more of the end and corner points coincide are not treated here, as the transformation mentioned does not apply to such cases. On the other hand this transformation carries the problem

with discontinuous integrand, the problem of Roos,¹ and others into problems of Bolza, as is shown in §7.

The problem to be studied in this paper is a generalization of the following problem studied by Miss Mary E. Sinclair.² When a pair of points, (x_1, y_1) and (x_2, y_2) , are given, then a curve $y = y(x)$ joining these points and a number $y(\xi)$, where $x_1 < \xi < x_2$, give to the expression $J = [y(\xi)]^2 + 2 \int_{x_1}^{x_2} y \sqrt{1 + y'^2} dx$ a definite value. The problem of minimizing J in a class of such sets $(\xi, y(x))$ is one which arose in connection with the classical minimum area problem. Hadamard³ treated the more general problem of minimizing $J = \varphi(\xi, \eta) + \int_{x_1}^{x_2} f(x, y, y') dx$, where $\eta = y(\xi)$, $x_1 < \xi < x_2$, by the device of transforming the problem into one of ordinary type but with discontinuous integrand. Later Clarke⁴ studied the same problem by direct methods. He found some additional necessary conditions and showed that, suitably strengthened, they are sufficient to insure a minimum.

In §1 of the present paper the problem to be studied is stated precisely, and in §2 it is transformed into a problem of Bolza which is proved equivalent to it. In §3-§6 the known results for the problem of Bolza are applied to the original problem. The concluding section is devoted to a survey of other problems which can be treated by similar methods.

¹C. F. Roos, A general problem of minimizing an integral with discontinuous integrand, Transactions of the American Mathematical Society, vol. 31 (1929), p. 58.

²Mary E. Sinclair, Concerning a compound discontinuous solution in the problem of the surface of revolution of minimum area, Annals of Mathematics, 2d series, vol. 10 (1909), p. 55.

³J. Hadamard, Lecons Sur le Calcul des Variations, Paris, 1910, p. 176.

⁴E. H. Clarke, The Minimum of the Sum of a Definite Integral and a Function of a Point, Doctoral Dissertation, University of Chicago, 1922.

1. Formulation of the Problem. Let x_a denote the set of variables $(x_0, x_1, \dots, x_q)$, which are always understood to satisfy the inequalities $x_0 < x_1 < \dots < x_q$; similarly let $y_1(x)$, or simply $y(x)$, denote the set of functions $(y_1(x), y_2(x), \dots, y_n(x))$, $x_0 \leq x \leq x_q$; and let $y_1(x_a)$ denote the set of $n(q+1)$ quantities obtained by letting $i = 1, \dots, n$ and $a = 0, \dots, q$ independently. Finally, let $y_1'(x) = dy_1(x)/dx$ ($i = 1, \dots, n$). Then the problem to be studied in this paper may be described as that of finding in a class of sets $[x_a, y(x)]$ satisfying the differential equations

$$(1:1) \quad \varphi_\beta [x, y, y'] = 0 \quad (\beta = 1, \dots, m < n),$$

and the end and corner conditions

$$(1:2) \quad \psi_\mu [x_a, y_1(x_a)] = 0 \quad (\mu = 1, \dots, p \leq (n+1)(q+1)),$$

a set which minimizes a functional J of the form

$$(1:3) \quad J = g[x_a, y_1(x_a)] + \int_{x_0}^{x_q} f[x, y, y'] dx.$$

In order to state the problem more explicitly, let R_1 be a region of $(2n+1)$ -dimensional points $[x, y, y']$ in which the functions φ_β and f have continuous partial derivatives of at least the third order, and in which the matrix of partial derivatives $\|\varphi_{\beta y_1'}\|$ has rank m . Similarly, let R_2 be a region of $[(n+1)(q+1)]$ -dimensional points $[x_a, y_1(x_a)]$ in which the functions ψ_μ and g have continuous partial derivatives of at least the third order, and in which the matrix of derivatives

$$\|\psi_{\mu x_a} \quad \psi_{\mu y_1(x_a)}\|$$

has rank p . An arc in R_1 is defined to be an arc $y_1(x)$, $x_0 \leq x \leq x_q$, which is continuous and consists of a finite number of

sub-arcs on each of which the functions $y_1(x)$ have continuous derivatives, and whose elements (x, y, y') are all interior to R_1 . An admissible arc is a set x_a and an arc in R_1 , all of whose end and corner elements $[x_a, y_1(x_a)]$ are interior to R_2 . It is easily seen that an admissible arc gives to the functions $f, \varphi_\theta, g, \psi_\mu$ and J well-defined values. The problem may now be stated as that of finding in the class of admissible arcs satisfying (1:1) and (1:2) one which minimizes the functional J of (1:3).

It should be noted that the statement of the problem excludes those minima, if any such exist, for which $x_b = x_{b+1}$, where b is some positive integer between 0 and $q - 1$. Such minima are, in a sense, singular cases, and the transformation given below of the problem into a problem of Bolza does not apply to these cases.

2. Transformation of the Problem. The functional J can be rewritten in the equivalent form

$$J = g[x_a, y_1(x_a)] + \sum_{b=0}^{q-1} \int_{x_b}^{x_{b+1}} f[x, y, y'] dx.$$

In the $(b + 1)$ -st term of this summation let the change of variable $x = x_b + (x_{b+1} - x_b)t$ be made, as the first step in the transformation. Let the set of functions $z_r(t)$ ($r = 1, \dots, qn$) be defined by the equations

$$z_{bn+1}(t) = y_1(x), \quad x = x_b + (x_{b+1} - x_b)t \\ (b = 0, \dots, q - 1; i = 1, \dots, n).$$

It follows immediately that

$$(2:1) \quad y_1(x_b) = z_{bn+1}(0), \quad y_1(x_{b+1}) = z_{bn+1}(1),$$

and that

$$(2:2) \quad z_{cn+1}(1) = z_{cn+n+1}(0) \quad (c = 0, \dots, q - 2).$$

As the next step, let the abscissas x_a be replaced by the $q + 1$ constant functions $z_{qn+a+1}(t)$ which must therefore satisfy the differential equations

$$(2:3) \quad z'_{qn+a+1}(t) = 0 \quad (a = 0, \dots, q).$$

Here and elsewhere it is understood that $z_j'(t) = dz_j(t)/dt$ ($j = 1, \dots, qn + q + 1 \equiv N$). In applying this transformation, from the variables x_a, x, y_1 to the variables t, z_j , to the function $g[x_a, y_1(x_a)]$, x_a is replaced by $z_{qn+a+1}(1)$ (cf. 2:3), $y_1(x_0)$ by $z_1(0)$ and $y_1(x_{b+1})$ by $z_{bn+1}(1)$, (cf. 2:1). There results a function of the variables $z_j(0), z_j(1)$, which is denoted by $g^*[z(0), z(1)]$. It is then seen that under the transformation the functional J becomes the functional

$$J^* = g^*[z(t_1), z(t_2)] + \int_{t_1}^{t_2} f^*(t, z, z') dt,$$

where $t_1 = 0, t_2 = 1$, and

$$(2:4) \quad f^*(t, z, z') = \sum_{b=0}^{q-1} f[x, y, y'](x_{b+1} - x_b);$$

in the $(b + 1)$ -st term of this summation $x = x_b + (x_{b+1} - x_b)t$, $y_1(x) = z_{bn+1}(t)$, and $y_1'(x) = z'_{bn+1}(t)/(x_{b+1} - x_b)$, and in each term x_a is replaced by $z_{qn+a+1}(t)$ for $a = 0, \dots, q$.

In a similar manner, let

$$\varphi_{bm+\rho}^*[t, z, z'] = \varphi_\rho[x, y, y'] \quad (\rho = 1, \dots, m; b = 0, \dots, q-1)$$

in which for each b the substitutions just described are applied to the right members. Also, let

$$\varphi_{qm+a+1}^*[t, z, z'] = z'_{qn+a+1}.$$

The functions φ_α^* ($\alpha = 1, \dots, qm + q + 1 \equiv M$) thus obtained are all defined on $0 \leq t \leq 1$, and are equal to zero by (1:1) and (2:3).

The end conditions associated with the transformed problem can be written in terms of new functions

$$\psi_\nu^*[t_1, z(t_1), t_2, z(t_2)] \quad (\nu = 1, \dots, (q-1)n + p + 2 \equiv P)$$

defined as follows: the function ψ_μ^* , for $\mu = 1, \dots, p$, is obtained from ψ_μ in exactly the manner by which g^* was obtained from g , (and hence depends only on $z(t_1)$ and $z(t_2)$); $\psi_{cn+1+p}^* \equiv z_{cn+1}(t_2) - z_{cn+n+1}(t_1)$ for $c = 0, \dots, q-2$; $\psi_{p-1}^* \equiv t_1$; and $\psi_p^* \equiv t_2 - 1$. The functions ψ_ν^* are equal to zero by (1:2) and (2:2), and the definitions of t_1 and t_2 .

The original problem has thus been transformed into the problem of finding, in a class of arcs defined by functions $z_j(t)$ satisfying $\varphi_\alpha^* = 0$ and $\psi_\nu^* = 0$, an arc which minimizes $J^* = g^*[z(t_1), z(t_2)] + \int_{t_1}^{t_2} f^*[t, z, z'] dt$. It remains to be shown that this problem satisfies all the hypotheses usually made for a problem of Bolza¹ and is equivalent to the original problem.

At this point it should be noted that the original problem could have been stated in terms of the region R_2 and the region R_0 , where R_0 is a set of $(q+2+2n)$ -dimensional points $[x_a, x, y_1, y_1']$, ($a = 0, \dots, q$; $1 = 1, \dots, n$), whose $[x, y_1, y_1']$ -coordinates lie in R_1 and whose x_a -coordinates are projections of points in R_2 . It will be shown that the transformation above, considered as a point transformation, carries sets of points of R_0 into a region R_0^* of $(q+2+2qn)$ -dimensional points $[x_a, t, z_r, z_r']$, ($r = 1, \dots, qn$). For a given set x_a there are q values of x which correspond to the same value t by means of the q equations $x = x_b + (x_{b+1} - x_b)t$. Any ordered set

¹G. A. Bliss, The Problem of Bolza in the Calculus of Variations, Mimeographed Lecture Notes, University of Chicago, 1935.

of q points in R_0 , all of which have the same x_a -coordinates, and whose x -coordinates are related in the way just described and are ordered according to increasing value, are called associated points. Let R_0^* be the set of points $[x_a, t, z_r, z_r']$ obtained by applying the q sets of equations $x = x_b + (x_{b+1} - x_b)t$, $y_1 = z_{bn+1}$, $y_1' = z_{bn+1}' / (x_{b+1} - x_b)$ ($b = 0, \dots, q-1$), respectively to the q points in a set of associated points. It is clear that this correspondence established between points of R_0^* and sets of points of R_0 is one-to-one and continuous, since by hypothesis $x_0 < x_1 < \dots < x_q$. More precisely, let $P_b \equiv (x_a, x^{(b)}, y_1^{(b)}, y_1^{(b)'})$ and $\bar{P}_b \equiv (\bar{x}_a, \bar{x}^{(b)}, \bar{y}_1^{(b)}, \bar{y}_1^{(b)'})$, ($b = 0, \dots, q-1$), be two sets of associated points in R_0 , transforming respectively into points $P^* \equiv (x_a, t, z_r, z_r')$ and $\bar{P}^* \equiv (\bar{x}_a, \bar{t}, \bar{z}_r, \bar{z}_r')$ of R_0^* . Then it is true that, for any $\epsilon > 0$, there exists a number δ such that the conditions

$$\begin{aligned} |\bar{x}_a - x_a| &< \delta, & |\bar{x}^{(b)} - x^{(b)}| &< \delta, \\ |\bar{y}_1^{(b)} - y_1^{(b)}| &< \delta, & |\bar{y}_1^{(b)'} - y_1^{(b)'}| &< \delta \end{aligned}$$

imply the conditions

$$|\bar{x}_a - x_a| < \epsilon, \quad |\bar{t} - t| < \epsilon, \quad |\bar{z}_r - z_r| < \epsilon, \quad |\bar{z}_r' - z_r'| < \epsilon.$$

Furthermore, from the definitions of the function f^* and the first qm of the functions φ_α^* ($\alpha = 1, \dots, qm + q + 1 = M$) it follows that in R_0^* they are continuous and have continuous partial derivatives of at least the third order. The remaining functions φ_α^* are linear in their arguments, being in fact the functions z_{qn+a+1}' . Hence if R_1^* is the region of points (t, z_j, z_j') ($j = 1, \dots, N$), obtained from R_0^* by replacing x_a by z_{qn+a+1} and adjoining the coordinates z_{qn+a+1}' , it follows that

in R_1^* all of the functions φ_α^* and f^* have the continuity properties just described. Further, the matrix $\|\varphi_{\alpha z_j}^*\|$ has rank M in R_1^* . For the definition of the functions φ_α^* shows that this matrix consists of blocks of which the non-diagonal ones are zero, the diagonal block of the last $q + 1$ rows and columns the identity matrix, and the other diagonal blocks are obtained as follows. The first qm functions φ_α^* can be written $\varphi_{bm+\beta}^*$ ($b = 0, \dots, q - 1$; $\beta = 1, \dots, m$). The diagonal blocks referred to above are the $m \times n$ blocks of elements $\varphi_{bm+\beta, z_{bn+1}}^*$ for each fixed value of b . These q blocks are obtained by applying the transformation to the q matrices $\|\varphi_{\beta y_1}/(x_{b+1} - x_b)\|$ on the q ranges $x_b \leq x \leq x_{b+1}$, respectively, so that each has rank m . Hence the matrix $\|\varphi_{\alpha z_j}^*\|$ has rank $qm + q + 1 = M$.

It is easily seen that the region R_2 of points $[x_a, y_1(x_a), y_1(x_{b+1})]$ is transformed in one-to-one, continuous manner into a region R_3^* of points $[z_{qn+a+1}(t_2), z_1(t_1), z_{bn+1}(t_2)]$, (cf. equations (2:1), (2:3)), and that in R_3^* the function g^* and the first p functions ψ_ν^* have continuous partial derivatives of at least the third order. The linearity of the remaining functions ψ_ν^* shows that in the region R_2^* of points $[t_1, z_j(t_1), t_2, z_j(t_2)]$, obtained by adjoining to R_3^* the necessary coordinates, the functions g^* and ψ_ν^* all have these continuity properties. In R_2^* it may be verified that the matrix $\|\psi_{\nu t_1}^*, \psi_{\nu z_j(t_1)}^*, \psi_{\nu t_2}^*, \psi_{\nu z_j(t_2)}^*\|$ has rank P .

Using the definition of admissible arcs for a problem of Bolza given by Bliss,¹ with R_1^* and R_2^* as the regions involved, the transformed problem may be stated as that of finding in the class of admissible arcs satisfying the equations $\varphi_\alpha^* = \psi_\nu^* = 0$

¹Bliss, op. cit., p. 10.

one which minimizes the functional J^* . Only those solutions for which the constant functions z_{qn+a+1} are all distinct will be considered in this paper. The transformation used above carries an admissible arc $[x_a, y_1(x)]$, satisfying $\varphi_\beta = \psi_\mu = 0$ and giving to J a certain value, into an admissible arc $[z_{qn+a+1}(t), z_r(t)]$ satisfying $\varphi_\alpha^* = \psi_\nu^* = 0$ and giving to J^* the same value; and conversely. Further, it carries neighboring arcs into neighboring arcs, so that the two problems are equivalent.

3. The Multiplier Rule. Normality. In this section the necessary condition for the problem of Bolza known as the multiplier rule is stated. It is then transformed to give a multiplier rule for the original problem. For the problem of Bolza an admissible solution E^* of the equations $\varphi_\alpha^* = 0$ ($\alpha = 1, \dots, M$), defined on the interval $t_1 t_2$, is said to satisfy the multiplier rule if there exist constants λ_ν^*, ℓ_ν ($\nu = 1, \dots, P$) not all zero and a function $F^*(t, z, z', \lambda^*) = \lambda_0^* f^* + \lambda_\alpha^*(t) \varphi_\alpha^*$ with multipliers $\lambda_\alpha^*(t)$ continuous on $t_1 t_2$ except possibly at values t defining corners of E^* where they have well-defined right and left limits, such that equations of the form

$$(3:1) \quad F_{z_j'}^* = \int_{t_1}^t F_{z_j}^* dt + c_j \quad (j = 1, \dots, N)$$

are satisfied along E^* , and furthermore such that the equation

$$(3:2) \quad [(F^* - z_j' F_{z_j'}^*), dt + F_{z_j}^* dz_j]_1^2 + \lambda_0^* dg^* + \ell_\nu d\psi_\nu^* = 0$$

holds at the ends of E^* for every choice of the differentials $dt_1, dz_{j1}, dt_2, dz_{j2}$. Every minimizing arc E^* for the problem of Bolza must satisfy the multiplier rule. It is understood that in equation (3:2), as elsewhere, the customary summation convention for repeated indices is observed.

The function F^* occurring in the equations above may be written

$$(3:3) \quad F^* = \lambda_0^* f_b^*(x, y, y')(x_{b+1} - x_b) + \lambda_{bm+\beta}^*(t) \varphi_{b\beta}^*(x, y, y') + \lambda_{qm+a+1}^* z'_{qn+a+1},$$

where $f_b^*(x, y, y') = f(x, y, y')$ with, for each b , the variable x replaced by $x_b + (x_{b+1} - x_b)t$, $y_1(x)$ by $z_{bn+1}(t)$, and x_a by $z_{qn+a+1}(t)$; and similarly for $\varphi_{b\beta}^*$. The first qn of the equations $F_{z_j}^* = \int_{t_1}^t F_{z_j}^* dt + c_j$ of (3:1), when F^* is replaced by its value from (3:3), have the form

$$(3:4) \quad \lambda_0^* f_{by_1}^* + \lambda_{bm+\beta}^* \varphi_{b\beta y_1}^* / (x_{b+1} - x_b) - \int_{t_1}^t \lambda_0^* f_{by_1}^*(x_{b+1} - x_b) dt - \int_{t_1}^t \lambda_{bm+\beta}^* \varphi_{b\beta y_1}^* dt - c_{bn+1} = 0$$

(b not summed; $b = 0, \dots, q-1$; $i = 1, \dots, n$; $\beta = 1, \dots, m$).

The transforms $\lambda_0, \lambda_\beta(x)$ of the multipliers $\lambda_0^*, \lambda_\alpha^*(t)$ ($\alpha = 1, \dots, M$) are defined by the equations

$$\lambda_0 = \lambda_0^*, \quad \lambda_\beta(x) = \lambda_{bm+\beta}^* \left(\frac{x - x_b}{x_{b+1} - x_b} \right) \cdot \frac{1}{x_{b+1} - x_b}.$$

The equations (3:4) then transform into the equations

$$(3:5) \quad \lambda_0 f_{y_1} + \lambda_\beta \varphi_{\beta y_1} - \int_{x_b}^x (\lambda_0 f_{y_1} + \lambda_\beta \varphi_{\beta y_1}) dx - c_{bn+1} = 0,$$

$$x_b \leq x \leq x_{b+1}.$$

If the function $F(x, y, y', \lambda)$ is defined by the equation

$F = \lambda_0 f + \lambda_\beta \varphi_\beta$ then the equations (3:5) become

$$(3:6) \quad F_{y_1} - \int_{x_b}^x F_{y_1} dx - c_{bn+1} = 0, \quad x_b \leq x \leq x_{b+1}.$$

Of the remaining $q + 1$ of the equations $F_{z_j}^* = \int_{t_1}^t F_{z_j}^* dt + c_j$, the first will suffice to illustrate the method of

transformation used. This is the equation

$$\begin{aligned}
 (3:7) \quad & \lambda_{qm+1}^* - \int_{t_1}^t \lambda_0^* f_{0x}^* (1-t)(x_1 - x_0) dt - \int_{t_1}^t \lambda_0^* z_1' f_{0y_1}^* \frac{dt}{x_1 - x_0} \\
 & + \int_{t_1}^t \lambda_0^* f_0^* dt - \int_{t_1}^t \lambda_\beta^* \varphi_{0\beta x}^* (1-t) dt \\
 & - \int_{t_1}^t \lambda_\beta^* \varphi_{0\beta y_1}^* \frac{z_1'}{(x_1 - x_0)^2} dt - c_{qn+1} = 0.
 \end{aligned}$$

This equation, unlike those in (3:4), does not by itself restrict the minimizing arc but merely determines the multiplier $\lambda_{qm+1}^*(t)$. This multiplier is associated with the differential equation $z'_{qn+a+1} = 0$ and has no significance for the original problem. However by combining equation (3:7) with two others, as shown below, a transversality condition for the original problem is found. The two equations needed are obtained by equating to zero the coefficients of $dz_{qn+1}(t_2)$ and $dz_{qn+1}(t_1)$ in the identity (3:2). They may be written

$$\begin{aligned}
 (3:8) \quad & \lambda_{qm+1}^*(t_2) + \lambda_0^* g_{z_{qn+1}}^*(t_2) + \ell_\mu \psi_{\mu z_{qn+1}}^*(t_2) = 0 \\
 & (\mu = 1, \dots, p), \\
 & \lambda_{qm+1}^*(t_1) = 0.
 \end{aligned}$$

Combining these equations with those obtained from (3:7) by putting, in turn, $t = t_1$ and $t = t_2$ gives

$$\begin{aligned}
 (3:9) \quad & \lambda_0^* g_{z_{qn+1}}^*(t_2) + \ell_\mu \psi_{\mu z_{qn+1}}^*(t_2) + \int_{t_1}^{t_2} \lambda_0^* f_{0x}^* (1-t)(x_1 - x_0) dt \\
 & + \int_{t_1}^{t_2} \lambda_0^* z_1' f_{0y_1}^* \frac{dt}{x_1 - x_0} - \int_{t_1}^{t_2} \lambda_0^* f_0^* dt \\
 & + \int_{t_1}^{t_2} \lambda_\beta^* \varphi_{0\beta x}^* (1-t) dt + \int_{t_1}^{t_2} \lambda_\beta^* z_1' \varphi_{0\beta y_1}^* \frac{dt}{(x_1 - x_0)^2} = 0 \\
 & (\mu = 1, \dots, p; i = 1, \dots, n; \beta = 1, \dots, m).
 \end{aligned}$$

When the arc E^* has no corners and has continuous second derivatives, the following equation holds:

$$(3:10) \quad \frac{d}{dt} \left[\lambda_0^* f_0^* + \lambda_\beta^* \varphi_{0\beta}^* \frac{1}{x_1 - x_0} - \lambda_0^* f_{0y_1}^* \frac{z_1'}{x_1 - x_0} - \lambda_\beta^* \varphi_{0\beta y_1}^* \frac{z_1'}{(x_1 - x_0)^2} \right] = \lambda_0^* f_{0x}^* (x_1 - x_0) + \lambda_\beta^* \varphi_{0\beta x}^*.$$

For, the left member of this equation, when differentiated, gives the right member, plus terms which cancel on account of (3:4), plus two terms in z_1'' which cancel each other. Denoting the bracketed expression in (3:10) by A , and substituting $\frac{dA}{dt}$ for $\lambda_0^* f_{0x}^* (x_1 - x_0) + \lambda_\beta^* \varphi_{0\beta x}^*$ in (3:9) gives an equation, one term of which is $\int_{t_1}^{t_2} \frac{dA}{dt} (1-t) dt$. Applying ordinary integration by parts to this term gives, since $\varphi_{0\beta}^* = 0$ along E^* , the equation

$$\begin{aligned} & \left[(\lambda_0^* f_0^* + \lambda_\beta^* \varphi_{0\beta}^* \frac{1}{x_1 - x_0} - \lambda_0^* f_{0y_1}^* \frac{z_1'}{x_1 - x_0} - \lambda_\beta^* \varphi_{0\beta y_1}^* \frac{z_1'}{(x_1 - x_0)^2}) (1-t) \right]_{t_1}^{t_2} \\ & + \lambda_0^* g_{z_{qn+1}}(t_2) + \ell_\mu \psi_{\mu z_{qn+1}}(t_2) = 0, \end{aligned}$$

which transforms into

$$\lambda_0 g_{x_0} + \ell_\mu \psi_{\mu x_0} - [\lambda_0 f + \lambda_\beta \varphi_\beta - y_1' (\lambda_0 f_{y_1} + \lambda_\beta \varphi_{\beta y_1})]^{x_0} = 0,$$

or the equivalent form

$$(3:11a) \quad \lambda_0 g_{x_0} + \ell_\mu \psi_{\mu x_0} - [F - y_1' F_{y_1}]^{x_0} = 0.$$

Similarly, at the corners x_{c+1} ($c = 0, \dots, q-2$) and the end-point x_q of the arc E the following conditions must hold:

$$(3:11b) \quad \begin{aligned} \lambda_0 g_{x_{c+1}} + \ell_\mu \psi_{\mu x_{c+1}} - [F - y_1' F_{y_1}]_{x_{c+1}}^{x_{c+1}+} &= 0, \\ \lambda_0 g_{x_q} + \ell_\mu \psi_{\mu x_q} - [F - y_1' F_{y_1}]_{x_q} &= 0. \end{aligned}$$

Equations (3:8) were obtained from the transversality conditions (3:2); the other equations obtainable from it will be considered next. They fall into three sets: (a) the coefficients of dt_1 and dt_2 set equal to zero, which are of no value as they merely determine the constants ℓ_{p-1} and ℓ_p associated with $\psi_{p-1}^* = t_1$ and $\psi_p^* = t_2 - 1$; (b) the coefficients of $dz_{qn+a+1}(t_2)$ and $dz_{qn+a+1}(t_1)$ ($a = 0, \dots, q$) set equal to zero, which are used in deriving equations (3:11b) just as (3:8) were used in deriving equation (3:11a); and (c), the coefficients of $dz_{bn+1}(t_2)$ and $dz_{bn+1}(t_1)$ ($b = 0, \dots, q-1$; $i = 1, \dots, n$) set equal to zero, which are to be studied next. The equations obtained in this set for $b = 0$ are used as examples. They are

$$(3:12) \quad \left[\lambda_0^* f_{0y_1}^* + \lambda_\beta^* \varphi_{0\beta y_1}^* \frac{1}{x_1 - x_0} \right]^{t_2} + \lambda_0^* g_{z_1}^*(t_2) + \ell_\mu \psi_{\mu z_1}^*(t_2) + \ell_{1+p} = 0,$$

$$(3:13) \quad - \left[\lambda_0^* f_{0y_1}^* + \lambda_\beta^* \varphi_{0\beta y_1}^* \frac{1}{x_1 - x_0} \right]^{t_1} + \lambda_0^* g_{z_1}^*(t_1) + \ell_\mu \psi_{\mu z_1}^*(t_1) = 0.$$

Equations (3:13) transform to give

$$(3:14a) \quad \lambda_0 g_{y_1}(x_0) + \ell_\mu \psi_{\mu y_1}(x_0) - [F_{y_1}]^{x_0} = 0.$$

For each i from 1 to n the equation (3:12), when added to the equation associated with $dz_{n+1}(t_1)$, gives (the constant ℓ_{1+p} cancelling) the equation

$$\lambda_0 g_{y_1}(x_1) + \ell_\mu \psi_{\mu y_1}(x_1) - [F_{y_1}]_{x_1}^{x_1^+} = 0.$$

Similarly, at the corner points x_{c+1} and the end-point x_q of the arc E the following conditions must hold:

$$\begin{aligned}
 (3:14b) \quad & \lambda_0 y_1(x_{c+1}) + \ell_\mu \psi_{\mu y_1}(x_{c+1}) - [y_{y_1}]_{x_{c+1}}^{x_{c+1}+} = 0, \\
 & \lambda_0 y_1(x_q) + \ell_\mu \psi_{\mu y_1}(x_q) - [F_{y_1}]_{x_q} = 0.
 \end{aligned}$$

These results may be combined with arguments about the coincident vanishing of λ_0 , ℓ_μ or the identical vanishing of λ_0 , $\lambda_\beta(x)$ to yield a multiplier rule. An admissible solution E of the equations $\varphi_\beta = 0$, defined on an interval $x_1 x_2$, is said to satisfy the multiplier rule if there exist constants λ_0 , ℓ_μ not all zero and a function $F(x, y, y', \lambda) = \lambda_0 f + \lambda_\beta(x) \varphi_\beta$ with multipliers $\lambda_\beta(x)$ continuous on $x_1 x_2$ except possibly at corners of E where they have well-defined right and left limits, such that equations (3:6), (3:11) and (3:14) are satisfied. For an arc E satisfying the multiplier rule the multipliers λ_0 , $\lambda_\beta(x)$ do not vanish simultaneously at any set of associated points on E. Every minimizing arc E must satisfy the multiplier rule.

To justify these statements it should be noted that if λ_0 , $\lambda_\beta(x)$ vanish simultaneously at a set of associated points as defined in §2, then λ_0^* , $\lambda_{bm+\beta}^*(t)$ all vanish for a value t on $0 \leq t \leq 1$. A consideration of the Euler-Lagrange equations, as written in terms of new variables by Bliss,¹ shows that $F_{z_1}^*$ all vanish identically on $0 \leq t \leq 1$, and that $F_{z_{qn+a+1}}^* = \lambda_{qm+a+1}^*$ are constants. The transversality conditions then show that λ_0^* , $\lambda_\alpha^*(t)$ all vanish at the given value t , which is not true for an arc satisfying the multiplier rule for a problem of Bolza. To show that $\lambda_0 = \lambda_0^*$, ℓ_μ cannot all vanish, consider the equation $\lambda_0^* J_1^*(\xi, \eta) + \ell_\nu \Psi_\nu^*(\xi, \eta) = 0$, used in proving the multiplier rule, which must hold for all sets of admissible

¹Op. cit., pp. 21 ff.

variations $\varepsilon_1, \varepsilon_2, \eta_j(t)$ satisfying along E^* the equations of variation $\Phi_\alpha^* = 0$ introduced in §5. By proper choice of the functions η_j it can be shown that the vanishing of λ_0^*, l_μ implies the vanishing of $\lambda_0^*, l_1, l_2, \dots, l_p$. The argument of Bliss¹ then shows that $\lambda_0^*, \lambda_\alpha^*(t)$ vanish identically, contrary to the statement of the multiplier rule.

An extremal for the problem of Bolza is an arc without corners in R_1^* together with a set of multipliers,

$$z_j(t), \lambda_0^*, \lambda_\alpha^*(t) \quad (j = 1, \dots, N; \alpha = 1, \dots, M; 0 \leq t \leq 1)$$

such that the functions $z_j'(t), \lambda_\alpha^*(t)$ have continuous derivatives and satisfy the equations (3:1) and $\varphi_\alpha^* = 0$. An extremal is called non-singular in case the determinant

$$R^* = \begin{vmatrix} F_{z_j' z_k'}^* & \varphi_{\alpha_1 z_j'}^* \\ \varphi_{\alpha_1 z_k'}^* & 0 \end{vmatrix} \quad (j, k = 1, \dots, N; \alpha_1, \alpha = 1, \dots, M)$$

is different from zero along it. An extremal is normal if it satisfies the multiplier rule with a unique set of multipliers having $\lambda_0^* = 1$. For the original problem an extremal is defined to be an arc in R_1 and a set of multipliers

$$y_1(x), \lambda_0, \lambda_\beta(x) \quad (i = 1, \dots, n; \beta = 1, \dots, m; x_0 \leq x \leq x_q)$$

having corners only at the values x_{c+1} ($c = 0, \dots, q-2$) for which the functions $y_1'(x), \lambda_\beta(x)$ have continuous derivatives between corners and satisfy the equations (3:6) and $\varphi_\beta = 0$.

An extremal is non-singular in case the determinant

$$R = \begin{vmatrix} F_{y_n' y_1'} & \varphi_{\gamma y_n'} \\ \varphi_{\beta y_1'} & 0 \end{vmatrix} \quad (n, i = 1, \dots, n; \beta, \gamma = 1, \dots, m)$$

¹Op. cit., pp. 26 ff.

is different from zero along it. Further, an extremal is normal if it satisfies the multiplier rule with a unique set of multipliers having $\lambda_0 = 1$. Evidently normality for the original problem implies normality for the transformed problem. Although those portions of the theory of the problem of Bolza which apply to abnormal arcs could be transformed for abnormal arcs for the problem considered in this paper, these cases will not be treated here. This restriction of normality for the original problem is one of convenience of notation, as it obviates the need for introducing later a reduced accessory minimum problem.

4. The Weierstrass and Clebsch Conditions. The second necessary condition on a minimizing arc E^* for the problem of Bolza is condition II^{*}, also known as the Weierstrass condition. It is stated in terms of the function $E^*(t, z, z', \lambda^*, Z')$ defined by the equation

$$E^*(t, z, z', \lambda^*, Z') = F^*(t, z, Z', \lambda^*) - F^*(t, z, z', \lambda^*) \\ - (Z_j' - z_j') F_{z_j'}^*(t, z, z', \lambda^*) \quad (j = 1, \dots, N).$$

An arc E^* is said to satisfy the Weierstrass condition if the condition

$$(4:1) \quad E^*(t, z, z', \lambda^*, Z') \geq 0$$

is valid at every element (t, z, z', λ^*) of the arc E^* for all sets $(t, z, Z') \neq (t, z, z')$ interior to the region R_1^* and satisfying the equations $\varphi_{\alpha}^* = 0$. Every normal minimizing arc for the problem of Bolza must satisfy this condition.

By equation (3:3) it follows that the function in (4:1) may be written

$$\begin{aligned}
 & E^*(t, z, z', \lambda^*, Z') \\
 &= \lambda_0^* [f_b^*(x, y, Y') - f_b^*(x, y, y')](x_{b+1} - x_b) \\
 &+ \lambda_{bm+\beta}^* [\varphi_{b\beta}^*(x, y, Y') - \varphi_{b\beta}^*(x, y, y')] \\
 (4:2) \quad & - (Z'_{bn+1} - z'_{bn+1}) [\lambda_0^* f_{by_1'}^*(x, y, y') \\
 &+ \lambda_{bm+\beta}^* \varphi_{b\beta y_1'}^*(x, y, y') \frac{1}{x_{b+1} - x_b}] \\
 & (b = 0, \dots, q-1; \beta = 1, \dots, m; i = 1, \dots, n),
 \end{aligned}$$

where in f_b^* and $\varphi_{b\beta}^*$ and their partial derivatives the substitutions $x = x_b + (x_{b+1} - x_b)t$, $y_1(x) = z_{bn+1}(t)$, $Y_1' = Z'_{bn+1}/(x_{b+1} - x_b)$ and $x_a = z_{qn+a+1}$ are to be made. The expression obtained from the right-hand member of (4:2) by fixing $b = b'$ must be non-negative in the subclass of sets (t, z, Z') which is obtained from the class of sets (t, z, Z') , satisfying the hypotheses of the Weierstrass condition, by setting $Z'_{bn+1} = z'_{bn+1}$ for every b except $b = b'$. For this subclass must satisfy the equations $\varphi_\alpha^* = 0$, namely the $q+1$ equations $z'_{qn+a+1} = 0$ and the q groups of equations $\varphi_{bm+\beta}^* = 0$ ($b = 0, \dots, q-1$), since each of these groups involves, of the variables z_j' , only the set z'_{bn+1} . Conversely, if each of the q expressions obtained by fixing b is non-negative in its respective subclass then the Weierstrass condition is satisfied. By transforming these q expressions for the original problem, it is readily seen that the q conditions

$$\begin{aligned}
 & (x_{b+1} - x_b) \{ \lambda_0 [f(x, y, Y') - f(x, y, y')] \\
 &+ \lambda_\beta [\varphi_\beta(x, y, Y') - \varphi_\beta(x, y, y')] \\
 &- (Y_1' - y_1') [\lambda_0 f_{y_1'}(x, y, y') + \lambda_\beta \varphi_{\beta y_1'}(x, y, y')] \} \geq 0
 \end{aligned}$$

must hold at every element (x, y, y', λ) of E on the q intervals $x_b \leq x \leq x_{b+1}$, respectively, for all sets $(x, y, Y') \neq (x, y, y')$ interior to R_1 and satisfying the equations $\varphi_\beta = 0$. The factor

$(x_{b+1} - x_b)$ is positive, and hence may be deleted. These q conditions and the definition of the function $F(x, y, y', \lambda)$ then show that the condition

$$F(x, y, y', \lambda, Y_1') = F(x, y, Y_1', \lambda) - F(x, y, y', \lambda) - (Y_1' - y_1') F_{y_1'}(x, y, y', \lambda) \geq 0$$

must hold at every element of E for the sets just mentioned. This is the second, or the Weierstrass, necessary condition for the original problem. It follows that every normal minimizing arc must satisfy the Weierstrass condition.

The Clebsch condition is a third necessary condition on a normal minimizing arc E . An arc E is said to satisfy the necessary condition of Clebsch if the condition

$$F_{y_h' y_1'}(x, y, y', \lambda) \pi_h \pi_1 \geq 0 \quad (h, i = 1, \dots, n)$$

is valid at every element (x, y, y', λ) of E for all sets $(\pi_1, \dots, \pi_n) \neq (0, \dots, 0)$ satisfying the equations

$$\varphi_{\beta y_1'}(x, y, y') \pi_1 = 0.$$

Every normal minimizing arc E must satisfy this condition. This can be proved in the usual way as a consequence of the Weierstrass condition for the original problem, or it can be proved equivalent to the Clebsch condition for the corresponding problem of Bolza.

5. Fourth Necessary Conditions. Before discussing the fourth necessary conditions used in this paper some further notations are needed. A set of admissible variations is a set of functions $\eta_j(t)$, $(0 \leq t \leq 1; j = 1, \dots, N)$, continuous and having continuous derivatives except at a finite number of points, together with a pair of constants ξ_1, ξ_2 . The equations of variation along E^{ib} are the equations

$$\Phi_{\alpha}^{*}(t, \eta, \eta') = \varphi_{\alpha z_j}^{*} \eta_j + \varphi_{\alpha z_j'}^{*} \eta_j' = 0, \\ (\alpha = 1, \dots, M; j = 1, \dots, N),$$

in which the arguments of the derivatives of φ_{α}^{*} are the functions $z_j(t)$ defining E^{*} . The equations of variation of the end conditions are the equations

$$\begin{aligned} \Psi_{\nu}^{*}(\xi_1, \eta(t_1), \xi_2, \eta(t_2)) &= (\psi_{\nu t_1}^{*} + z_{j1}' \psi_{\nu z_{j1}}^{*}) \xi_1 \\ &+ \psi_{\nu z_{j1}}^{*} \eta_j(t_1) + (\psi_{\nu t_2}^{*} + z_{j2}' \psi_{\nu z_{j2}}^{*}) \xi_2 \\ &+ \psi_{\nu z_{j2}}^{*} \eta_j(t_2) = 0 \quad (\nu = 1, \dots, p) \end{aligned}$$

in which the arguments of the derivatives of ψ_{ν}^{*} are the end-values of E^{*} . If E^{*} is a normal non-singular extremal satisfying the equations $\psi_{\nu}^{*} = 0$ then for every set of admissible variations $\xi_1, \xi_2, \eta_j(t)$ satisfying the equations $\Phi_{\alpha}^{*} = 0, \Psi_{\nu}^{*} = 0$ along E^{*} there exists an admissible one-parameter family $z_j(t, b)$, $0 \leq t \leq 1$, of arcs satisfying the equations $\varphi_{\alpha}^{*} = 0, \psi_{\nu}^{*} = 0$, containing E^{*} for the value $b = 0$, having $\xi_1, \xi_2, \eta_j(t)$ for its variations along E , and having the continuity properties described by Bliss.¹ The second variation calculated for this family has the form

$$J^{**}(0) = 2q^{*} + \mathcal{L}_{\nu} 2q_{\nu}^{*} + \int_{t_1}^{t_2} 2\omega^{*}(t, \eta, \eta') dt$$

where $2q^{*}$ is the quadratic form in $\xi_1, \eta_j(t_1), \xi_2, \eta_j(t_2)$, whose coefficients are the second derivatives of g^{*} , $2q_{\nu}^{*}$ is the corresponding form for the function ψ_{ν}^{*} , and $2\omega^{*}$ is the quadratic form

$$2\omega^{*}(t, \eta, \eta') = F_{z_j z_k}^{*} \eta_j \eta_k + 2F_{z_j z_k'}^{*} \eta_j \eta_k' + F_{z_j' z_k'}^{*} \eta_j' \eta_k'.$$

¹Bliss, op. cit., p. 51.

The accessory minimum problem is the problem of finding in the class of sets of admissible variations $\xi_1, \xi_2, \eta_j(t)$ satisfying the equations $\Phi_\alpha^* = 0, \Psi_\nu^* = 0$ along E^* one which minimizes the function $J^{**}(0)$ above. The equations of the extremals for this problem are the equations

$$(5:1) \quad \frac{d}{dt} \Omega_{\eta_j}^* - \Omega_{\eta_j}^* = 0, \quad \Phi_\alpha^* = 0$$

where Ω^* is the function

$$\Omega^*(t, \eta, \eta', \mu^*) = \omega^* + \mu_\alpha(t) \Phi_\alpha^*.$$

The equations (5:1) will next be put in canonical form. Consider the subset of the equations $\zeta_j = \Omega_{\eta_j}^*, 0 = \Phi_\alpha^*$ which may be written in the form

$$\begin{aligned} \zeta_{bn+1} &= \Omega_{\eta_{bm+1}}^*(t, \eta, \eta', \mu), \\ 0 &= \Phi_{bm+\beta}^*(t, \eta, \eta') \\ (b &= 0, \dots, q-1; i = 1, \dots, n; \beta = 1, \dots, m). \end{aligned}$$

For a fixed value of b only the variables $\eta_{bn+1}, \eta_{qn+a+1}, \eta'_{bn+1}, \mu_{bm+\beta}$ occur in the right-hand members of these equations; from this fact and the definitions of Φ_α^*, Ω^* it follows that the functional determinant of these equations, for any fixed value of b , with respect to $\eta'_{bn+1}, \mu_{bm+\beta}$ is the determinant

$$\begin{vmatrix} (r_{by_h' y_1'}^* + \frac{\lambda_{bm+\beta}^*}{x_{b+1} - x_b} \varphi_{b\beta y_h' y_1'}^*) \frac{1}{x_{b+1} - x_b}, & \frac{\varphi_{b\gamma y_1'}^*}{x_{b+1} - x_b} \\ \frac{\varphi_{b\beta y_h'}^*}{x_{b+1} - x_b} & 0 \end{vmatrix};$$

that this determinant is non-zero follows from the hypothesis of non-singularity of the arc E^* . Hence the equations above may be solved to yield the following equations:

$$\eta'_{bn+i} = \pi^*_{bn+i}(t, \eta_{bn+h}, \eta_{qn+a+1}, \zeta_{bn+h}),$$

$$\mu_{bm+\beta} = M^*_{bm+\beta}(t, \eta_{bn+h}, \eta_{qn+a+1}, \zeta_{bn+h}).$$

The other set of the equations $\zeta_j = \Omega^*_{\eta_j}, 0 = \Phi^*_\alpha$ is

$$\begin{aligned} \zeta_{qn+a+1} &= \mu_{qn+a+1} \equiv M^*_{qn+a+1}, \\ (5:2) \quad 0 &= \eta'_{qn+a+1} \equiv \pi^*_{qn+a+1} \quad (a = 0, \dots, q); \end{aligned}$$

these equations are already in solved form. The canonical accessory equations for the problem of Bolza then are the equations

$$\eta'_j = \pi^*_j, \quad \zeta'_j = \Omega^*_{\eta_j} \quad (j = 1, \dots, N),$$

where in the function $\Omega^*_{\eta_j}$ the arguments η_j' are replaced by

π^*_j and the arguments μ_α by M^*_α . In terms of the function

$\mathcal{H}^*(t, \eta, \zeta)$, defined by the relation $\mathcal{H}^* = \zeta_j \pi^*_j -$

$\Omega^*(t, \eta, \pi^*, M^*)$, these equations become $\eta'_j = \mathcal{H}^*_{\zeta_j}, \quad \zeta'_j = -\mathcal{H}^*_{\eta_j}$.

To transform the various functions used above, let new functions $u_s(x), v_i(x)$ ($s = 1, \dots, n + q + 1; i = 1, \dots, n$), be defined by the equations

$$u_1(x) = \eta_{bn+1}(t), \quad x = x_b + (x_{b+1} - x_b)t,$$

$$u_{n+a+1}(x) = \eta_{qn+a+1}(t),$$

$$v_1(x) = \zeta_{bn+1}(t), \quad x = x_b + (x_{b+1} - x_b)t,$$

so that the functions $u_{n+a+1}(s)$ are constants. It should be noted that $\mathcal{H}^*$ does not depend on the functions ζ_{qn+a+1} , as is readily seen from the definition of Ω^* and equations (5:2).

Further, $\mathcal{H}^*$ is a sum of functions each containing, of the variables $\eta_{bn+i}, \zeta_{bn+i}$, only those for a fixed b ; this follows from the form of Ω^* and the distribution of arguments in the functions π^*, M^* . Hence $\mathcal{H}^*$ transforms readily into a function

$\mathcal{H}(x, u_s, v_1)$ defined on the sub-arcs of E . Since $u_1'(x) = \eta'_{bn+1}(t)/(x_{b+1} - x_b)$, and so on, the equations $\eta'_{bn+1} = \mathcal{H}^*_{\xi_{bn+1}}$, $\eta'_{qn+a+1} = \mathcal{H}^*_{\xi_{qn+a+1}} = 0$; $\xi'_{bn+1} = -\mathcal{H}^*_{\eta_{bn+1}}$ ($b = 0, \dots, q-1$; $a = 0, \dots, q$; $i = 1, \dots, n$) transform into $u_1' = \mathcal{H}_{v_1} \cdot (x_{b+1} - x_b)$, $u'_{n+a+1} = 0$, and $v_1' = -\mathcal{H}_{u_1} \cdot (x_{b+1} - x_b)$, respectively, on the sub-arcs of E . Consider, finally, the equations $\xi'_{qn+a+1} = -\mathcal{H}^*_{\eta_{qn+a+1}}$. The right members, in general, involve variables η_{bn+1} for different values b ; hence in the transformed equations $v'_{n+a+1} = -\mathcal{H}_{u_{n+a+1}} \cdot (x_{b+1} - x_b)$ the values of x on the sub-arcs of E are linearly related, that is,

$$v_{n+a+1}(x) = \xi_{qn+a+1}(t), \quad x = x_b + (x_{b+1} - x_b)t.$$

The canonical accessory equations for the original problem are then the equations $u_s' = \mathcal{H}_{v_s} \cdot (x_{b+1} - x_b)$, $v_s' = -\mathcal{H}_{u_s} \cdot (x_{b+1} - x_b)$, ($s = 1, \dots, n+q+1$).

Before showing how these equations are utilized in the fourth necessary conditions, two remarks are in place. The first is, that by transforming the functions Φ_α^* , Ψ_ν^* , ω^* , etc., formulas can be obtained whereby all the functions associated with the second variation of the original problem can be written down directly. These details are omitted in this paper for lack of space. The second remark is that the functions $u_1(x)$ defined above are continuous if they satisfy the transforms of the equations $\Psi_{cn+i+p}^* = 0$, which give $\eta_{cn+1}(t_2) = \eta_{cn+n+1}(t_1)$.

The end conditions $\Psi_\nu^* = 0$ for the accessory minimum problem are functions of $\xi_1, \xi_2, \eta_j(t_1), \eta_j(t_2)$. If the general solution of the canonical accessory equations is substituted in the equations $\Psi_\nu^* = 0$ a class $\xi_{1\theta}, \xi_{2\theta}, \eta_{j\theta}(t), \xi_{j\theta}(t)$ ($\theta = 1, \dots, T$) of linearly independent constants and solutions of these equations is determined. Let this class be restricted

to contain only sets $\xi_1, \xi_2, \eta_j, \zeta_j$ linearly independent of those sets for which $\xi_1 = 0, \xi_2 = 0, \eta_j(t) \equiv 0$. The value of the second variation for the set $\xi_1 = \xi_{1\theta}d_\theta, \xi_2 = \xi_{2\theta}d_\theta, \eta_j = \eta_{j\theta}d_\theta, \zeta_j = \zeta_{j\theta}d_\theta$ is a homogeneous quadratic form in the constants d_θ which is denoted by $Q^*(d)$. The transformation of $Q^*(d)$ for the original problem presents no difficulties, as the coefficients of d_θ outside the integral sign depend only on the end-values of $\xi_1, \xi_2, \eta_j, \zeta_j$ and the expression under the integral sign is a sum of q terms, each involving only the variables $\eta_{bn+1}, \eta'_{bn+1}$ (for a fixed value of b) and the constant functions η_{qn+a+1} . The transform is a quadratic form in d_θ , called $Q(d)$. The condition IV_3 will be stated in terms of this quadratic form and in terms of conjugate points defined as follows. For the problem of Bolza a point P on E^* , whose t -coordinate is designated by t_P , is conjugate to the initial point of E^* if there exists a solution η_j, ζ_j of the canonical accessory equation having $\eta_j(t_1) = \eta_j(t_P) = 0$ and having functions $\eta_j(t)$ not all identically zero on $t_1 \leq t \leq t_P$, and if there is no solution $\bar{\eta}_j, \bar{\zeta}_j$ of the canonical accessory equations with $\bar{\eta}_j(t) \equiv \eta_j(t)$ on $t_1 \leq t \leq t_P$ and $\bar{\eta}_j(t) \equiv 0$ on $t_P \leq t \leq t_2$. An equivalent definition may be obtained by restricting the functions η_j, ζ_j and $\bar{\eta}_j, \bar{\zeta}_j$ to be solutions only of the reduced set

$$(5:3) \quad \eta_j' = H_{\zeta_j}^*, \quad \zeta_r' = -H_{\eta_r}^* \\ (j = 1, \dots, qn + q + 1 = N; r = 1, \dots, qn).$$

For the variables ζ_{qn+a+1} do not appear in the canonical equations, and in particular are absent from the right-hand members of the equations $\zeta'_{qn+a+1} = -H_{\eta_{qn+a+1}}^*$. These variables enter in the canonical equations only where they are substituted for

the multipliers μ_{qn+a+1} (cf. equations (5:2)) associated with the functions η'_{qn+a+1} , so that the statement made is easily verified. Hence any solution of the canonical accessory equations defines a solution of (5:3), and conversely. Since the functions ξ_{qn+a+1} , ξ'_{qn+a+1} do not occur elsewhere in the two definitions of conjugate points, these definitions are equivalent. For the original problem a point $x^{(b)}$ of E on the interval $x_b x_{b+1}$ is defined to be a conjugate point to the point x_b of E in case there is a solution u_s , v_i of the reduced set of canonical accessory equations

$$(5:4) \quad \begin{aligned} u_s' &= \mathcal{H}_{v_s}, & v_i' &= -\mathcal{H}_{u_i} \\ & & (s &= 1, \dots, n+q+1; i = 1, \dots, n) \end{aligned}$$

on $x_b x_{b+1}$, having $u_s(x_b) = 0 = u_s(x^{(b)})$ and having functions $u_s(x)$ not all identically zero on $x_b x^{(b)}$, and such that there is no solution $\bar{u}_s$, $\bar{v}_i$ of (5:4) with $\bar{u}_s(x) \equiv u_s(x)$ on the interval $x_b x^{(b)}$ and with $\bar{u}_s(x) \equiv 0$ on $x^{(b)} x_{b+1}$. The set of associated points to a conjugate point $x^{(b)}$, namely the q points of E including $x^{(b)}$ whose x -coordinates correspond to the same value t under the transformation, transforms into a conjugate point for the problem of Bolza. For the solutions $u_s(x) \equiv 0$, $v_i(x) \equiv 0$ of (5:4), on all the intervals except $x_b x_{b+1}$, plus the set $u_s(x)$, $v_i(x)$ which determines the conjugate point $x^{(b)}$, together transform into a set $\eta_j(t)$, $\xi_r(t)$ defining a conjugate point P. This results from the fact that for the problem of Bolza the reduced canonical accessory equations fall into sets, each containing only variables η , ξ which transform into functions on the same sub-interval of E, plus variables η_{qn+a+1} which are constant by $\mathcal{H}_{\xi_{qn+a+1}}^* = \eta'_{qn+a+1} = 0$ and zero by the definition of conjugate point. It follows that a conjugate point for the

original problem determines one for the problem of Bolza, and the converse is evidently true.

An arc E^* , satisfying the hypotheses assumed above for the calculation of the second variation, is said to satisfy the necessary condition IV_3^* for the problem of Bolza if it has on it no point conjugate to the initial point, and if the condition $Q^*(d) \geq 0$ holds for all sets of constants d . Every normal non-singular minimizing arc without corners must satisfy the condition IV_3^* . Further, a normal non-singular extremal arc E which satisfies the equations $\psi_\mu = 0$ is said to satisfy the condition IV_3 if it has on it no point on a sub-arc conjugate to the initial point of that sub-arc and if the condition $Q(d) \geq 0$ holds for all sets of constants d . Every normal non-singular minimizing arc, with corners only at x_{c+1} , must satisfy the condition IV_3 . This statement is, in fact, merely a summary of the results of the preceding paragraphs.

Another formulation of the fourth necessary condition is given next. Consider the accessory boundary value problem, namely the problem of finding functions η_j , ξ_j and constants σ , ξ_1 , ξ_2 , ϵ_ν satisfying the differential equations and end-conditions

$$\begin{aligned} \eta_j' &= H^* \xi_j, & \xi_j' &= -H^*_{\eta_j} - \sigma \eta_j, \\ (5:5) \quad \xi_j d\eta_j]_1^2 + dq^* + \ell_\mu dq_\mu^* + \epsilon_\nu d\psi_\nu^* &\equiv 0, & \psi_\nu^* &= 0 \\ & (j = 1, \dots, N; \mu = 1, \dots, p; \nu = 1, \dots, P) \end{aligned}$$

the next to last equation being an identity in $d\xi_1$, $d\eta_{j1}$, $d\xi_2$, $d\eta_{j2}$. The elements of a fundamental system of solutions of the differential equations in (5:5) are functions of t and σ and the general solution involves $2N$ constants c_ρ linearly. The end conditions in (5:5) are $2N + 2 + P$ in number and are linear and

homogeneous in the $2N + 2 + P$ constants c_p , ξ_1 , ξ_2 , ϵ_ν . The determinant of coefficients in these end-conditions is a function $D^*(\sigma)$ which must vanish if this boundary value problem is to have a solution. The roots of $D^*(\sigma)$ which correspond to solutions of the boundary value problem having $\eta_j(t) \neq 0$ on the interval t_1, t_2 are called the characteristic numbers of the boundary value problem. An arc E^* is said to satisfy the condition IV_4^* if every characteristic number σ of the accessory boundary value problem along E^* satisfies the condition $\sigma \geq 0$. Every normal non-singular minimizing arc without corners must satisfy the condition IV_4^* . The elements of the determinant $D^*(\sigma)$ depend only on the values at t_1 and t_2 of the functions $\eta_j(t, \sigma)$, $\xi_j(t, \sigma)$ so that the usual transformation gives a determinant $D(\sigma)$. The roots of $D(\sigma)$ corresponding to solutions of the transformed boundary value problem having functions $u_j(x)$ not all identically zero on at least one of the intervals x_b, x_{b+1} are called the characteristic numbers for the arc E . An arc E , satisfying the hypotheses presupposed for the definition of the condition IV_3 , is said to satisfy the condition IV_4 if every characteristic number σ associated with E satisfies the condition $\sigma \geq 0$. Every normal non-singular minimizing arc, with corners only at x_{c+1} , must satisfy the condition IV_4 .

6. Sufficient Conditions. It has now been shown that, under certain assumptions, conditions I, II, III, IV_3 and IV_4 are necessary conditions on a minimizing arc. The remainder of this section is devoted to transforming for the original problem the sufficiency theorem for the problem of Bolza stated below.

An arc E with the properties described in the definition of condition II above is said to satisfy the strengthened condition II_N' if the inequality

$$E(x, y, y', \lambda, Y') > 0$$

is satisfied at every element (x, y, y', λ) satisfying the equations $\varphi_\beta = 0$ in a neighborhood N of those elements belonging to E and for all sets $(x, y, Y') \neq (x, y, y')$ in R_1 and satisfying $\varphi_\beta = 0$. The condition III' is defined to be the condition III above with the equality sign excluded. The condition IV₃' is defined to be the condition IV₃ plus the condition that the quadratic form $Q(d)$ be positive definite. Finally, the condition IV₄' is defined to be the condition IV₄ plus the condition that every solution $u_s(x)$, $v_s(x)$ of the canonical accessory equations which satisfies the end and transversality conditions have functions $u_s(x)$ which are identically zero on x_0x_1 . The conditions II_N*, III*, IV₃*, and IV₄* for the problem of Bolza are the transforms of conditions II_N', III', IV₃', and IV₄', respectively. It should be noted that the condition II_N' excludes from consideration the problem of Bolza dealing with solutions whose slopes are discontinuous, which otherwise could be included in the problem dealt with in this paper. It also excludes the exceptional case in which at a corner of E there is a set $(x, y, Y') \neq (x, y, y')$, in R_1 and satisfying the equations $\varphi_\beta = 0$, for which the E -function vanishes.

The non-tangency condition for the problem of Bolza which occurs in the statement of the theorem below is defined to be the condition that the matrix $\|\psi_{\nu\xi_1}^* \quad \psi_{\nu\xi_2}^*\|$ of coefficients of ξ_1 and ξ_2 in the functions ψ_ν^* shall have rank 2. For the problems of Bolza considered in this paper the non-tangency condition is always satisfied, since among the functions ψ_ν^* occur $\xi_1 - 0$ and $\xi_2 - 1$.

The following sufficiency theorem is proved by Bliss:¹

Let E^* be an admissible arc without corners satisfying the equations $\varphi_\alpha^* = \psi_\nu^* = 0$ for a problem of Bolza, and satisfying with a set of multipliers $\lambda_0^* = 1, \lambda_\alpha^*(t), \ell_\nu$ either the set of conditions $I^*, II_N^{*'}, III^{*'}, IV_3^{*'}$ or the set $I^*, II_N^{*'}, III^{*'}, IV_4^{*'}$ together with the non-tangency condition. Then there exists a neighborhood $\mathcal{F}^*$ of E^* in tz -space and a neighborhood $\mathcal{N}^*$ of the ends of E^* in the space of points $(t_1, z_{j1}, t_2, z_{j2})$ such that $J^*(C^*) > J^*(E^*)$ for every admissible arc C^* in $\mathcal{F}^*$ with ends in $\mathcal{N}^*$ satisfying the equations $\varphi_\alpha^* = \psi_\nu^* = 0$ and not identical with E^* . By transforming this theorem for the normal case the following sufficiency theorem for the original problem is obtained. Let E be a normal non-singular extremal arc with corners only at the distinct values x_{c+1} , satisfying the equations $\psi_\mu = 0$ and in addition either the set of conditions I, II_N', III', IV_3' or the set I, II_N', III', IV_4' . Then there exists a neighborhood $\mathcal{F}$ of E in xy -space and a neighborhood $\mathcal{N}$ of the ends and corners of E in the space of points $(x_a, y_1(x_a))$ such that $J(C) > J(E)$ for every admissible arc C in $\mathcal{F}$ with ends in $\mathcal{N}$ satisfying the equations $\varphi_\beta = \psi_\mu = 0$ and not identical with E .

7. Further Applications of the Transformation. Several generalizations of the problem studied in the preceding pages can be transformed into problems of Bolza in a similar manner. One such generalization occurs when, in the problem formulated in § 1, the arguments of the functions φ_β and of the function f are the variables $x, y_1(x), y_1'(x), x_a, y_1(x_a)$ ($i = 1, \dots, n; a = 0, \dots, q$). The transformation of § 2 can be applied to this problem,

¹op. cit., pp. 96 ff.

and the result can easily be put in the form of the problem of Bolza. Under the transformation the variables x_a occurring in the transforms of the functions φ_ρ and f are replaced by $z_{qn+a+1}(t)$, while the values $y_1(x_a)$ are expressed in terms of the end values of the functions $z(t)$ by the equations $y_1(x_0) = z_1(0)$, $y_1(x_{b+1}) = z_{bn+1}(1)$ ($b = 0, \dots, q-1$). The presence of these end values in the integrand and the differential equations is then eliminated by substitution of constant functions $z_{N+an+i}(t)$, where $N = qn + q + 1$, according to the equations

$$(7:1) \quad z_{N+1}(t) = z_1(0), \quad z_{N+(b+1)n+1}(t) = z_{bn+1}(1).$$

The functions $z_{N+an+i}(t)$ are thus restricted to satisfy the differential equations and end conditions

$$(7:2) \quad z'_{N+an+i}(t) = 0,$$

$$(7:3) \quad z_{N+1}(0) = z_1(0), \quad z_{N+(b+1)n+1}(1) = z_{bn+1}(1).$$

The problem of Bolza obtained by the substitutions (7:1) and by the adjunction, to the equations $\varphi_\alpha^* = 0$, $\psi_\nu^* = 0$, of (7:2) and (7:3) respectively is easily seen to be equivalent to the problem from which it was obtained.

A problem essentially different from the one just described may be formulated as follows: Let there be given two functions, $f_1(x, y, y')$ and $f_2(x, y, y')$, and a variable abscissa x_0 . The problem is that of finding in a class of arcs $y_1(x)$, $x_1 \leq x \leq x_2$, satisfying two sets of differential equations

$$\varphi_{1\alpha}(x, y, y') = 0 \quad (\alpha = 1, \dots, m_1; x_1 \leq x \leq x_0)$$

$$\varphi_{2\beta}(x, y, y') = 0 \quad (\beta = 1, \dots, m_2; x_0 \leq x \leq x_2)$$

and the end-conditions

$$\psi_{\mu}(x_1, y_1(x_1), x_0, y_1(x_0), x_2, y_1(x_2)) = 0 \quad (\mu = 1, \dots, p)$$

one which minimizes the functional

$$J = g(x_1, y_1(x_1), x_0, y_1(x_0), x_2, y_1(x_2)) \\ + \int_{x_1}^{x_0} f_1(x, y, y') dx + \int_{x_0}^{x_2} f_2(x, y, y') dx.$$

The transformation of §2 applied to the two intervals x_1x_0 and x_0x_2 carries this problem into a problem of Bolza. It is clear that the transformation could also be applied to the extended problem having any finite number of integrands and with some or all of the intervals of integration disjointed. It should be noted that the problem just described includes the problem with discontinuous integrand treated by Bliss and Mason¹ as well as the problem studied by Roos² and the still more general problem of Pixley.³ Finally, the generalizations involved in the two problems described in this section could be combined in a single problem to which the transformation is applicable.

The problem with solutions actually discontinuous can be similarly transformed. The resulting problem is or is not a problem of Bolza according as the comparison arcs of the original problem are or are not restricted in the manner indicated below.

To be specific, consider the problem of minimizing a functional $J = \int_{E_1}^{E_2} f(x, y, y') dx$ in a class of arcs $y_1(x)$, $E_1 \leq x \leq E_2$;

¹Bliss and Mason, A problem of the calculus of variations in which the integrand is discontinuous, Transactions of the American Mathematical Society, vol. 7 (1906), p. 325.

²C. F. Roos, A general problem of minimizing an integral with discontinuous integrand, Transactions of the American Mathematical Society, vol. 31 (1929), p. 58.

³H. H. Pixley, Discontinuous solutions in the problem of depreciation and replacement, American Journal of Mathematics, vol. 52 (1930), p. 851.

$i = 1, \dots, n$, having a finite number of discontinuities, and satisfying the equations $y_1(\xi_1) = A_1$, $y_1(\xi_2) = B_1$. For a given arc $y(x)$ with $q - 1$ discontinuities at $x = x_{c+1}$ ($c = 0, \dots, q-2$), define $x_0 = \xi_1$ and $x_q = \xi_2$. The transformation of §2 applied to the intervals $x_b x_{b+1}$ ($b = 0, \dots, q-1$) gives the functional

$$J^* = \int_{t_1}^{t_2} f_b(x, y, y')(x_{b+1} - x_b) dt,$$

where $f_b(x, y, y') = f(x, y, y')$ with, for each b , the usual substitutions $x = x_b + (x_{b+1} - x_b)t$, $y_1(x) = z_{bn+1}(t)$, $y_1'(x) = z'_{bn+1}(t)/(x_{b+1} - x_b)$, and $x_a = z_{qn+a+1}$ made. The functions $z_j(t)$ ($j = 1, \dots, qn + q + 1 = N$) are thus subject to the differential equations

$$(7:4) \quad z'_{qn+a+1} = 0 \quad (a = 0, \dots, q)$$

and the end conditions

$$(7:5a) \quad z_1(t_1) - A_1 = 0, \quad z_{qn-n+1}(t_2) - B_1 = 0,$$

where

$$(7:5b) \quad t_1 = 0, \quad t_2 - 1 = 0.$$

Conversely, given a positive integer q and the function f , then the number N of functions $z_j(t)$, the functional J^* , and so on, are determined, and hence the functional J . Each set of functions $z_j(t)$ satisfying (7:4) and (7:5) determines a set $\xi_1, \xi_2, y_1(x)$ with $q - 1$ (or fewer) points of discontinuity and such that $y_1(\xi_1) = A_1$, $y_1(\xi_2) = B_1$. Hence the problem of minimizing J^* in a class of arcs $z_j(t)$ satisfying (7:4) and (7:5) is equivalent to the original problem. When for the original problem, the number q is allowed to vary from arc to arc the transformed problem is not a problem of Bolza. If in the original problem it is desired merely to find conditions on an arc $y(x)$ that it minimize

the functional J in the class of arcs having at most the same number of discontinuities as $y(x)$, and in arbitrarily small neighborhoods, respectively, of those of $y(x)$, then the transformed problem is clearly a problem of Bolza. This last case, with $q = 1$, is the case chiefly studied in the past. In the first case the transformed problem, for q fixed, yields necessary conditions on a minimizing arc, as the class of arcs obtained by fixing q is a subclass of the comparison arcs.

